Gravity, Two-Body Problem

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Apps on Physics by Walter Fendt[1]

Abstract
Two-body problems are solved by reducing it into a single body and forming a central force problem. In this app, the motion of a two-body system is simulated with the origin at the center of mass of the two masses. We have utilized this app to produce data for various input parameters to generate different trajectories. One simple application of Sun-Earth system is shown using the app. This app provides the users with pictures and data to properly understand the concepts.

1 Aim of the experiment

To look at the motion of the two point-masses from the barycenter as origin and apply **Apps on Physics** in the following cases:

1. To understand the dynamics of a two-body problem
2. Verify the Kepler's third law
3. Simulate the Earth-Sun system as a two-body problem

2 Introduction

2.1 Two-Body Problem

The two-body problem can be described as the kinematics and dynamics of two point masses moving in the field produced by each other. Kepler studied this problem while observing planets and moons moving in the sky. Solving a two-body problem implies that given the initial values of the motion at $t = 0$, we need to find the position and other quantities at a time t.

The simplest way to solve this kind of problem is to consider the motion of the center of mass and treat it as a point in a central force field. Then the problem reduces to a central force problem, whose solution is well-known to scientists. We will not solve the Lagrangian equation of motions here due to its complexity, but there are certain results that are peculiar to a central force problem.

i. The motion always takes place in a plane.

ii. Angular momentum $l = mr^2\dot{\theta}$ is conserved. Which will be used further to prove Kepler's second law.

iii. The system is conservative and total energy is conserved.

Further mathematical calculations reveal that for gravitational force which follows the inverse square law i.e. $f \propto \frac{1}{r^2}$ the orbits are closed.* We can find the equation of orbit [2]:

$$\frac{1}{r} = C[1 + e\cos(\theta - \theta')] \tag{1}$$

where e is the eccentricity of the conic, and given by:

$$e = \sqrt{1 + \frac{2El^2}{mk^2}} \tag{2}$$

Thus various possible orbits seen are in the table 1.

Eccentricity	Energy	Orbit Type
$e > 1$	$E > 0$	Hyperbola
$e = 1$	$E = 0$	Parabola
$e < 1$	$E < 0$	Ellipse
$e = 0$	$E = \frac{-mk^2}{2l^2}$	Circle

Table 1: Types of orbit due to various values of eccentricity.

2.2 Kepler's Laws

Kepler, by observing the motions of celestial bodies, gave three empirical laws between 1609 to 1619 which were later on derived by classical mechanics. The laws are the following [3]:

1. **The Law of Orbits:** All planets move in elliptical orbits with the Sun at one of the foci. This is because we assume the mass of the Sun is very large compared to the planets $M >> m$. The eccentricity of Earth's orbit is 0.0167.
2. **The Law of Areas:** : A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals; that is, the rate $\frac{dA}{dt}$ at which it sweeps out area A is constant. This can be seen from the conservation of angular momentum as $\frac{dA}{dt} = \frac{r^2\dot{\theta}}{2} = \frac{l}{2m} = Constant.$
3. **The Law of Periods:** The square of the period of any planet is proportional to the cube of the semi-major axis of its orbit. Mathematically, $T^2 \propto r^3$ in planetary motion. This can be derived from the equation of orbit equation 1, but we will be skipping the mathematics.

2.3 Two-Body Problems in Nature

In nature, many problems are reduced to two-body problems for simplicity. The most prominent cases are seen in astronomy where planetary motions are studied using this framework. In our solar system, the motion of planets with respect to the sun is simplified to a two-body problem and solved accordingly. Earth-Sun and Earth-Moon systems are examples of this type. [4] Three-body problems do not have exact solutions and often lead to chaos. Thus we can only solve two-body problems exactly.

*Bertrand's theorem explains the derivation. In nature, only the forces where it is proportional to r^{-2} or r have closed orbits. $f \propto r$ is seen in elasticity.

3 Methodology

3.1 Setting-Up the Simulation

The Apps on Physics[1] provides us the app *Gravity, Two-Body Problem* to simulate the motion of two bodies under their gravitational fields in the center-of-mass frame. The interface of the app can be seen in the figure 1.

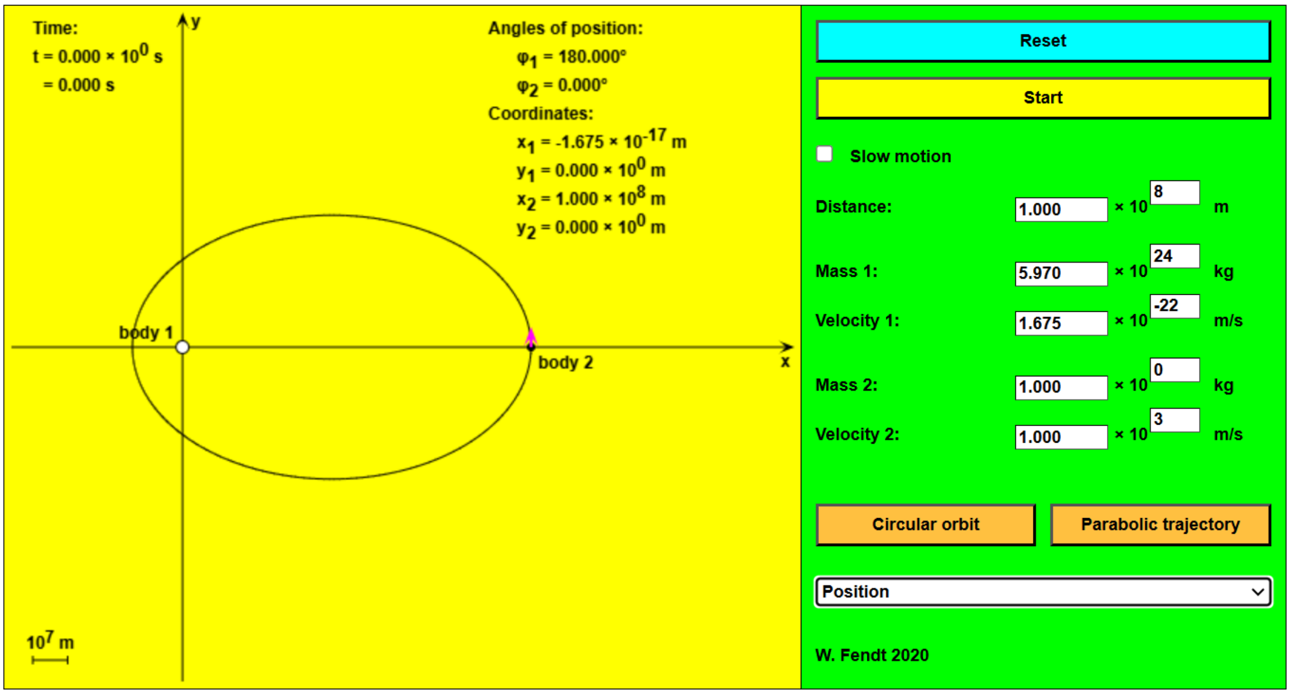

Figure 1: Interface of the *Gravity, Two-Body Problem* app showing the input and output parameters on the right hand side and the trajectory on the left.

It has the following inputs and outputs:

Input Parameters: Initial distance between two bodies, mass and initial velocity of body 1, mass and initial velocity of body 2.

Output Parameters: Orbit data, Position, Distance, Velocity, Acceleration, Force, Energy, Angular momentum, Orbital period.

We have the options to choose the path we want to see such as circular, parabolic, and elliptic. The app sets the velocities of mass 1 automatically to have the corresponding path. On top left corner time is simulated. At the bottom left corner, the scale of the plot is given. We have taken different values of velocity and mass to check their trajectory along with other parameters.

3.2 Output Parameters and Data Collection

For a fixed distance of $10^8 m$ we have varied the initial parameters and tabulated the various outputs produced by the app. With the available data in our hand we have used the orbital data and time periods of the motion to prove Kepler's third law. We have drawn a comparison with the semi-major axis and orbital period data.

Using the observable data of mass and initial velocities of earth and sun, we have simulated the Earth-Sun system and compared the simulated values with the available observed data. The initial parameters we have used are given as:

- Distance between the two bodies $= 1.496 \times 10^{11} m$
- Mass of the Sun $M_S = 1.988 \times 10^{30}$kg and Velocity $V_S = 8.946 \times 10^{-2}$m/s
- Mass of the Earth $M_E = 5.972 \times 10^{24}$kg and Velocity $V_E = 2.978 \times 10^{4}$m/s

We can choose different parameters to observe by selecting them at the box given at bottom right of figure 1.

4 Results

4.1 Motion in a Two-Body System

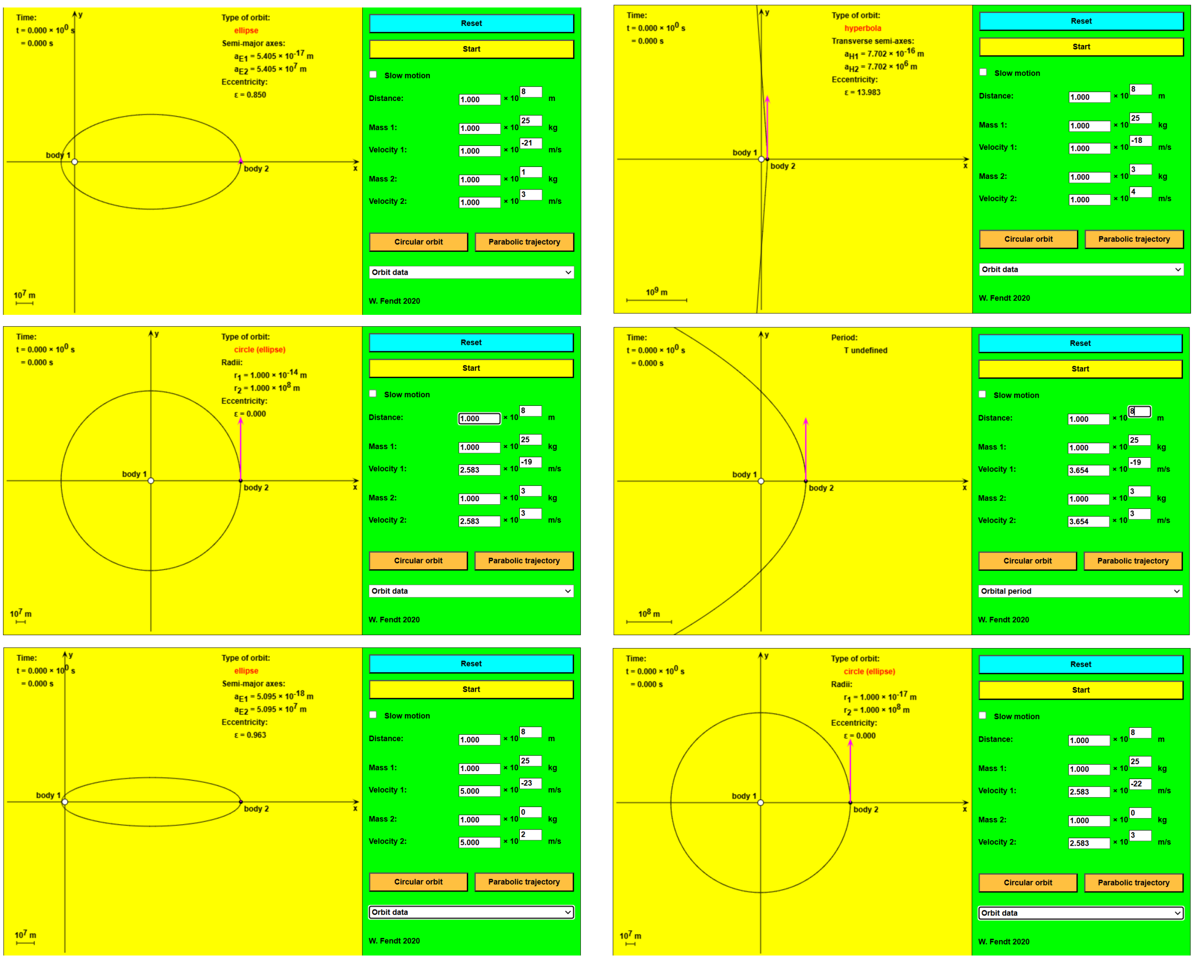

Figure 2: Various paths are shown in the figure for different values of initial parameters. Elliptic, parabolic, circular, and hyperbolic paths are seen and can be deduced from the eccentricity relation in table 1.

We have taken the mass of the first body to be 1×10^{25} for all observations and have kept the initial distance between the bodies as 10^8m. For various values of initial velocity and mass of

Sl No.	M_1(in kg)	V_1 (in m/s)	M_2(in kg)	V_2(in m/s)	Semi-major axis (in m)	Eccentricity	Orbital Period
1	1×10^{25}	1×10^{-21}	1×10^{1}	1×10^{3}	5.405×10^{7}	0.850	1.119 d
2	1×10^{25}	1×10^{-18}	1×10^{3}	1×10^{4}	7.702×10^{6}	13.983	-
3	1×10^{25}	2.583×10^{-19}	1×10^{3}	2.583×10^{3}	1.000×10^{8}	0	2.815 d
4	1×10^{25}	3.654×10^{-19}	1×10^{3}	3.654×10^{3}	2×10^{8}	1	-
5	1×10^{25}	5×10^{-23}	1×10^{0}	5×10^{2}	5.095×10^{7}	0.963	1.024 d
6	1×10^{25}	2.583×10^{-22}	1×10^{0}	2.583×10^{3}	1×10^{8}	0	2.815 d

Table 2: Data of output parameters for given values of mass and initial velocity of the two bodies.

the bodies we have plotted the following table 2. We can see from the above table that for different initial velocities the trajectories can be different. Here we have all the cases of possible trajectories. Orbital period is also tabulated where the orbits having hyperbolic and parabolic trajectories have no defined time period as they do not follow periodic motion. There are options to measure velocity, acceleration, etc. at any point along with the direction of motion. We have not utilized this option to keep the table simple.

4.2 Kepler's Law of Periods

Kepler's third law, the law of periods, can mathematically be written as,

$$T^2 \propto r^3 \implies \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3} = const. \tag{3}$$

where;
T = orbital time period
r = distance of semi-major axis
Now we can verify this empirical law using the values we have obtained from the simulation. Putting the values of row 1 and 5 from table 2 in the equation 3 we find,

$$\frac{T_1^2}{T_2^2} = \frac{1.119^2}{1.024^2} = 1.194$$
$$\frac{r_1^3}{r_2^3} = \frac{5.405^3}{5.095^3} = 1.194$$

Hence, we can see the Kepler's third law is verified by observation.

4.3 The Earth-Sun System

The simulation of the Earth-Sun system is done as a two-body problem using the app. We have used the initial parameters given in *section 3.2* and found the following results: We can see the type of orbit is given as an ellipse in figure 3 but with an eccentricity of 0. This is because the eccentricity of the Earth-Sun system is very close to zero and can be approximated to a circle. The time period of the orbit is found exactly to be 1 year; hence we can say the simulation is correct. Similarly, we can simulate other planet and sun systems using the app.

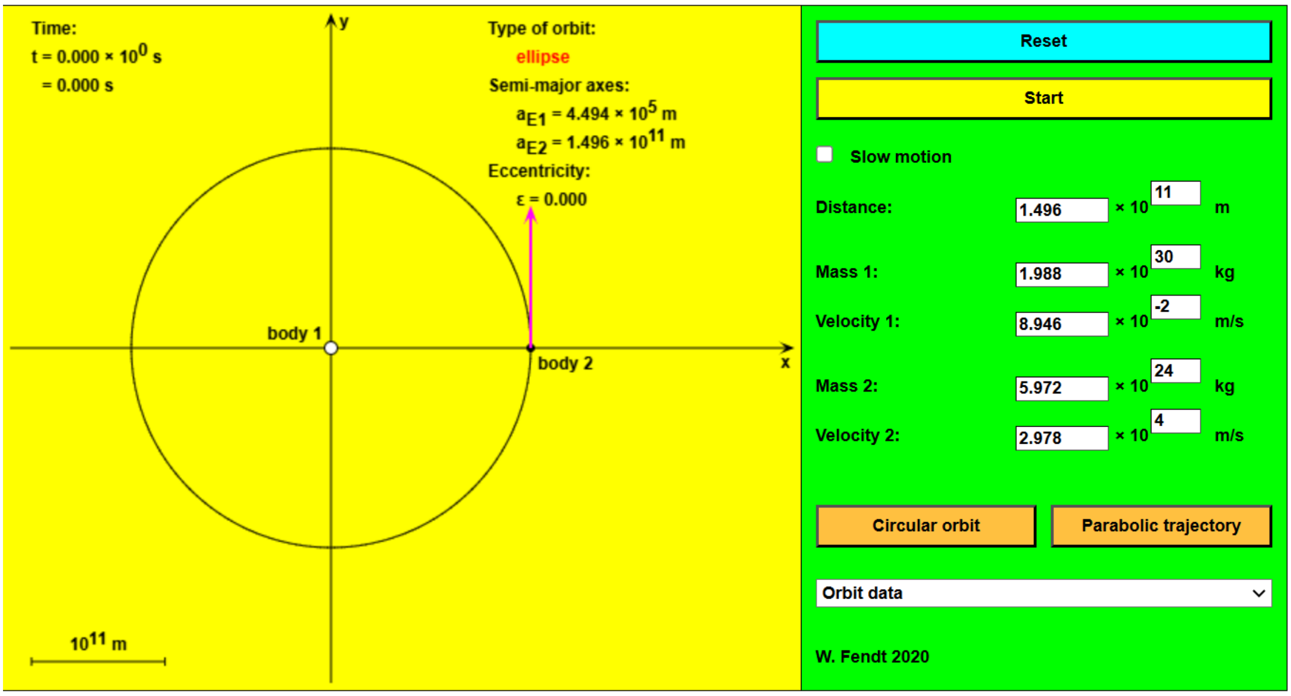

Figure 3: Earth-Sun system as a two-body problem as seen from the center-of-mass frame. The orbit is approximated to a circle and the semi-major axis has a distance 1.496×10^{11}m.

5 Conclusions

Scientists have developed a theoretical framework considering two-body problems to help them in astronomy. Kepler's laws jump-started these studies with having significance till now. Even though an infinite number of bodies are present in space we can approximate motions in sun-planet systems as two-body problems. This brings in the importance to study the theory and understand the applications deeply. We have used simple examples to illustrate the concept.

Apps on Physics[1] has this tool to simulate the motion of two-body problem and various parameters related to it. This app is useful in understanding and having a hands-on demonstration of the theory.

References

[1] Walter Fendt. Apps on Physics.

[2] H. Goldstein. *Classical Mechanics*. Pearson, 2011.

[3] D. Halliday, R. Resnick, and J. Walker. *Fundamentals of Physics*. Fundamentals of Physics. Wiley, 2013.

[4] H.C. Verma. *Concepts of Physics, Volume 1*. Bharati Bhawan Publishers & Distributors, Patna, India, reprint edition, 2022.