Inner and Outer Soddy Circles

Puja Solaiappan

Department of Mathematics
National Institute of Technology Rourkela,
Rourkela Odisha(Orissa) India,
769008

1 Introduction

Given are any three arbitrary non-collinear points A, B, and C. A set of three tangential circles can be drawn using A, B, and C as centres. Furthermore, two circles can be drawn that are tangential to the above three circles simultaneously in the interior and exterior sense. These are known as inner and outer Soddy circles.

The Soddy circles are very useful constructions and set a foundation for delving deeper into interesting engineering applications of surprisingly simple geometric constructions. These set of five mutually tangential circles with six common tangent points and tangent lines forms an upper limit i.e. no other non-intersecting circles can be drawn tangential to all three circles other than the inner and outer Soddy circles.

In this project, you will use GeoGebra to:

- Create a set of three mutually tangential circles given any three points to be used as centers of the circles.
- Create the inner and outer Soddy circles of the three mutually tangential circles.
- Learn the skills needed to construct the Apollonian gasket.

2 Construction

The first step of the procedure is constructing a triangle. The vertices of this triangle will represent the centers of three mutually tangential circles which are subsequently constructed. Next the Soddy circles are drawn - one within the delta gap and the other encompassing all three reference circles on the outside. The steps are as follows:

Step 1: Choose Polygon and click anywhere on the graph to create points A, B, C. Then click on point A again to form a closed triangle t1.

Step 2: Choose Angular Bisector. Create angular bisectors of the angles A, B and C of the reference triangle.

Step 3: Mark the intersection of the angular bisectors as K.

Step 4: Drop a perpendicular from K to the side AB.

Step 5: Mark the intersection of the perpendicular with the side AB as D.

Step 6: Using circle with Center Passing through a Point, create circles with center A passing through D and center B passing through D.

Step 7: Name the circle with center A as J_B and that with center B as J_B

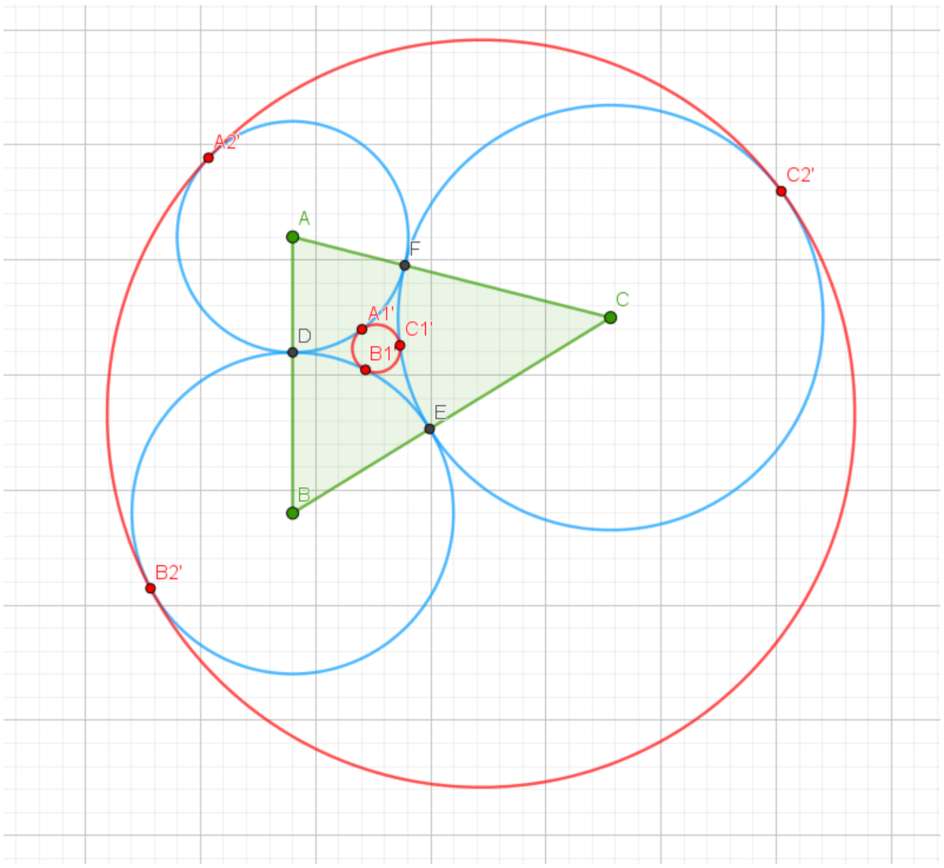


Figure 1: Inner and Outer Soddy Circles of a reference triangle ABC

Step 8: Mark the point of intersection of J_A and the side CA as F.

Step 9: Mark the point of intersection of J_B and the side BC as E.

Step 10: Draw a circle with center at C and passing through E. Name this circle J_C

Step 11: Draw a line through A and perpendicular to BC.

Step 12: The above line intersects the circle J_A at two points. Name the closer point A_2 and farther point A_1.

Step 13: Draw perpendicular lines to sides CA and AB through B and C respectively and in a similar manner define points B_1, B_2, C_1, C_2

Step 14: Plot lines EA_1 and EA_2.

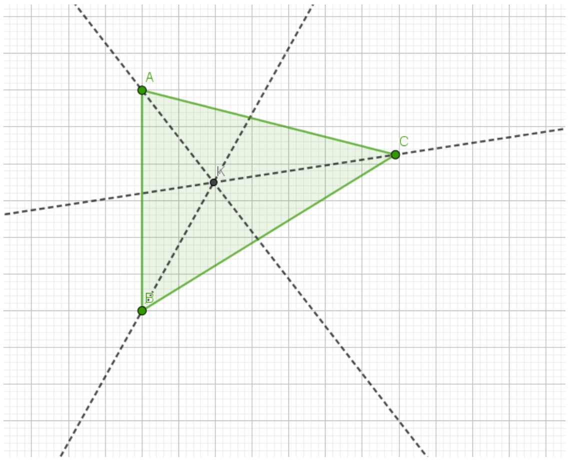
Step 15: Let the line EA_1 and EA_2 intersect the circle J_A again at A'_1 and A'_2.

Step 16: Similarly plot lines EC_1, EC_2, FB_1, FB_2. Define points B'_1, B'_2, C'_1, C'_2 in a similar fashion.

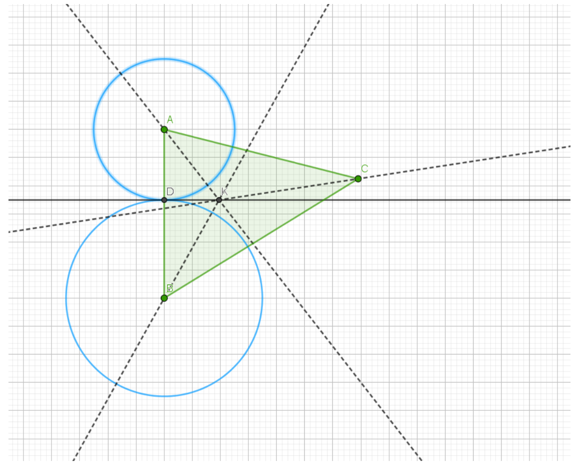
Step 17: Using Circle through 3 Points draw circles passing through $A'_1 - B'_1 - C'_1$ and $A'_2 - B'_2 - C'_2$. These are the inner and outer Soddy circles respectively.

Step 18: Use a show construction checkbox to select and deselect visibility options of the lines used in the construction.

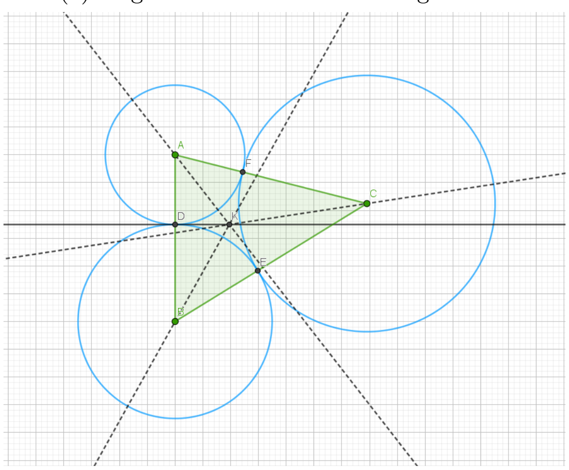
3 Geogebra Figures



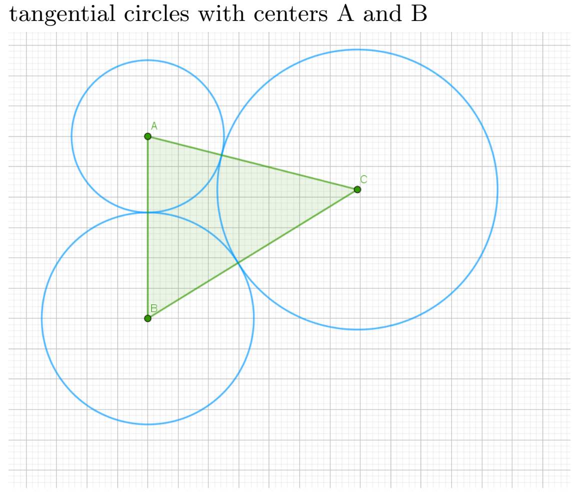
(a) Angular bisectors of the triangle ABC



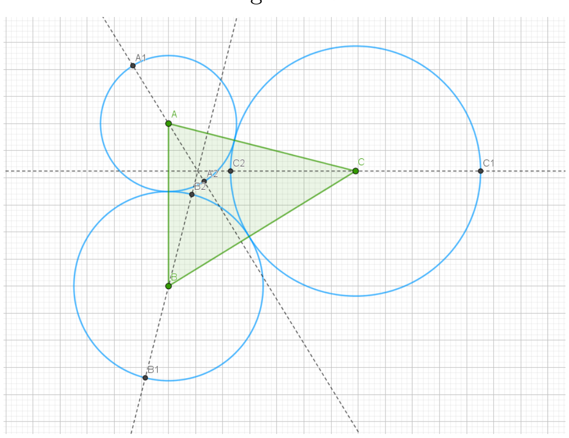
(b) Perpendicular dropped from K to side AB and tangential circles with centers A and B



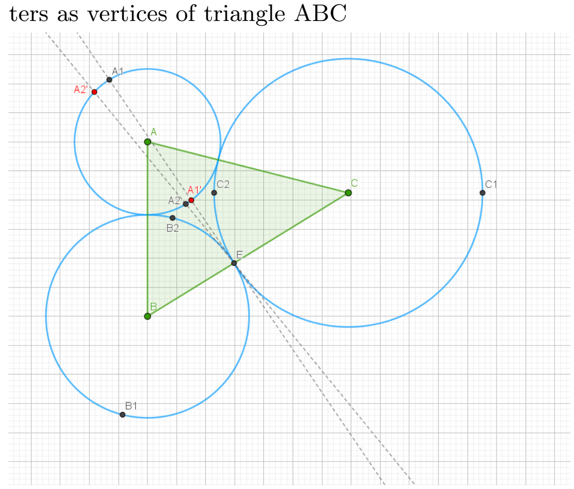
(c) Construction of three tangential circles with centers as vertices of triangle ABC



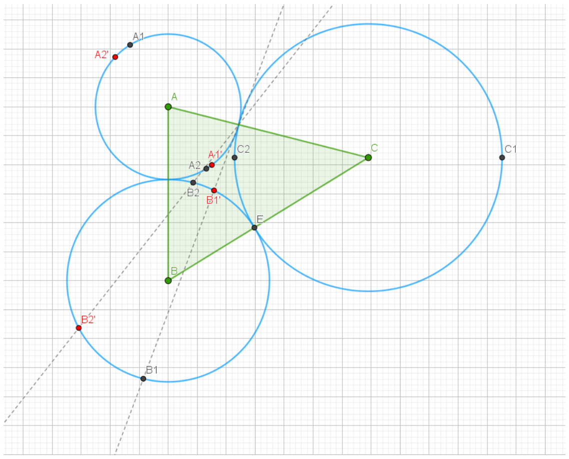
(d) Construction of three tangential circles with centers as vertices of triangle ABC



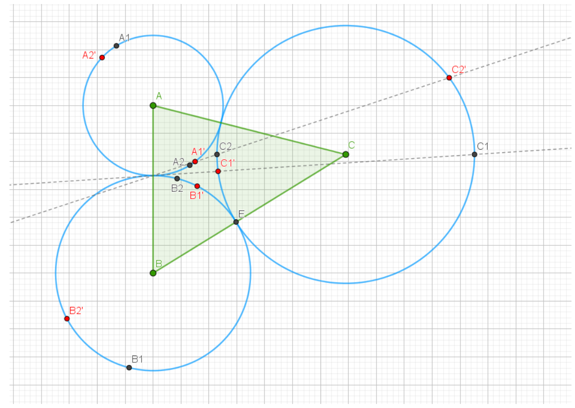
(e) Intersection points $A_1, A_2, B_1, B_2, C_1, C_2$ of perpendiculars with the three tangential circles



(f) Plotting points A_1' and A_2'



(g) Plotting points B_1', B_2'



(h) Plotting points C_1', C_2'

Figure 2: Steps to create Soddy circles in Geogebra

4 Experimenting

- Adjusting the positions of the vertices A, B, C automatically changes the radii and relative positions of the tangential and Soddy circles. For an equilateral triangle, the inner and outer Soddy circles are concentric.

- The signed values of curvatures of the circles are given by $\epsilon_i = \pm\frac{1}{r_i}$ where r gives the radius, $i = 1, 2, 3$ corresponds to the tangential circles and $i = 4$ corresponds to the Inner/Outer Soddy circle. Signs are taken as positive if the contact is external and negative if the contact is internal. Frederick Soddy (1936) gave an explicit formula relating the curvatures:

$$(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^2 = 2(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2)$$

  A negative value of ϵ_4 gives the curvature of the outer Soddy circle whereas a positive value gives the curvature of the inner Soddy circle.

- When Inner Soddy Circles are repeatedly created to fill in the delta gap of the tangential circles, a figure known as the Apollonian Gasket is formed. The construction involves the same steps as mentioned above but continuously redefines the three reference circles to include the Soddy circle recently created. Using the centers of the newly referenced circles, a new reference triangle is defined and is subsequently used for the construction of more inner Soddy Circles. A GeoGebra file has been attached to show the Apollonian Gasket formed by such repeated constructions.

5 Conclusion

The above procedure is the simplest method to construct the inner and outer Soddy circles. Understanding and applying the steps of this construction gives a solid foundation for developing the skills of advanced geometry using knowledge of tangency, curvature, and basic straight lines.