Triangle Centers

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Introduction

In this project, the basic centers of a triangle (incenter, circumcenter, centroid, orthocenter, nine-point circle, symmedian point, etc.) are constructed from their foundational definitions using GeoGebra. A comprehensive list of centers of a triangle was introduced by Clark Kimberling.

Geometric constructions of centers of triangles and associated theorems have been studied since Pythagoras. It is no surprise therefore that the literature of triangles is very rich and immensely rewarding to delve into. Triangles have been defined with many centers each equally important depending on what aspect you intend to focus on. A comprehensive list of centers of a triangle was introduced by Clark Kimberling. His list consists of more than 72,000 entries and gives a standard number to each center. You can check the encyclopedia here. We will be looking at the most common triangle centers in this project.

In this GeoGebra project, we will use the mathematical graphics software to:

- Create a triangle using matrices and the Polygon tool on the 2D graph.
- Plot the incenter by constructing angular bisectors for each angle of the triangle.
- Plot the circumcenter by constructing perpendicular bisectors for the triangle.
- Plot the centroid by constructing medians of the triangle.
- Plot the orthocenter by dropping perpendiculars from each vertex of the triangle to the opposite side.
- Sketch the incircle, circumcircle, and orthocircle i.e. nine-point circle.
- Reflect medians over corresponding angle bisectors to obtain the symmedian point.

Creating a Triangle

Step 1: Under the View tab open Algebra, Spreadsheet and Graphics2. (Note that this can also be done in 3D Graphics on the x-y-z plane). Adjust sizes of the sections according to your preference.

Step 2: In the spreadsheet, type the following matrix:

$$\begin{bmatrix} -3 & 1 & 0 \\ 4 & -2 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

These will serve as starting values for the vertices of your triangle. Each row represents the x, y, and z coordinates of a point. (Since we are working in the x-y plane, the z coordinates in defined as 0.)

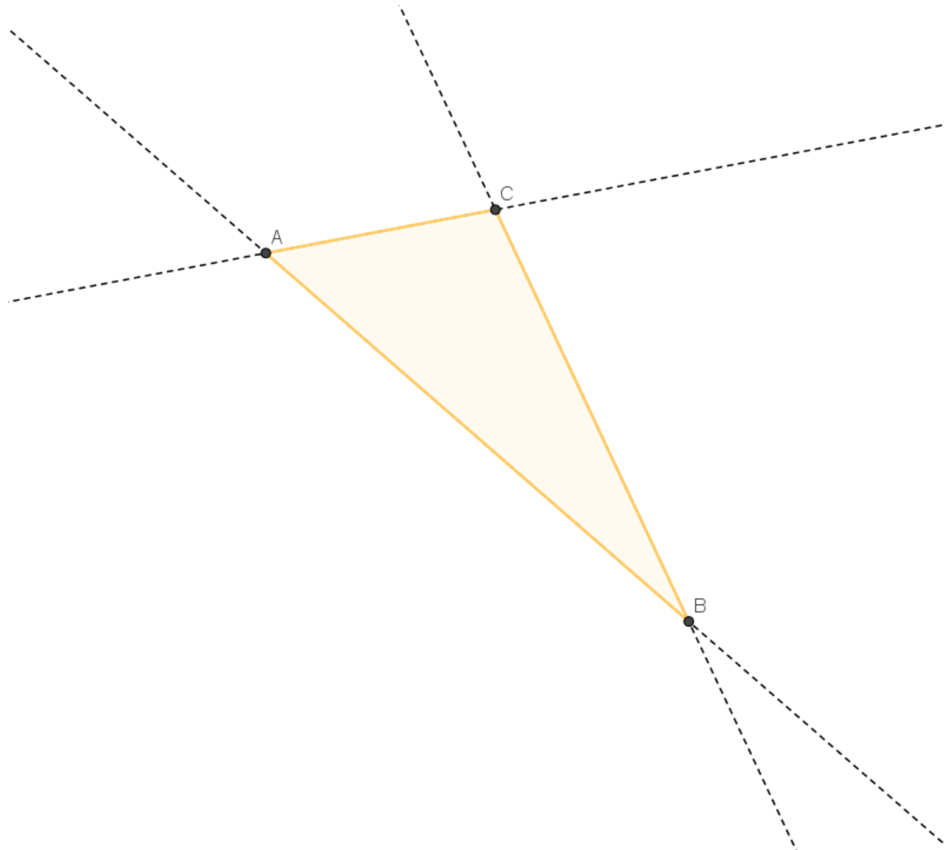

Figure 1: Constructing a Triangle ABC

Step 3: Select this 3x3 matrix in the spreadsheet and left click. Hover over the Create option then choose List of Points. This prompts the commands to be displayed in the algebra view and the graph. Notice that they are named A, B, C by default. You may click and drag the points wherever required in the graph and the algebra tab will continuously modify and display the current coordinates. A list l1 is also created, it can be deleted.

Step 4: The points A, B, and C can also be created by choosing the Point option and clicking on the desired three locations on the graph.

Step 5: Click on the graphics tab. Click on the square button with a triangle image. The bottom right of the button has a drop down arrow, choose Polygon.

Step 6: Click on B, A, C, then B again. A shaded triangle is formed.

Step 7: We will be using side lengths, usually written as a, b, and c, for some formulas. We will therefore rename a as SegBC, b as SegCA, and c as SegAB.

Step 8: In the input bar at the bottom of the screen, type the following commands to define lengths a, b, and c.

$$\text{a} = \text{Length(SegBC)}$$
$$\text{b} = \text{Length(SegCA)}$$
$$\text{c} = \text{Length(SegAB)}$$

Step 9: The semiperimeter is defined as half the perimeter. Define s = (a + b+ c)/2.

Step 10: Choose the third square icon with the line image and extend the line segments by clicking on the endpoints. Rename the lines as lineAB, lineBC, and lineCA.

Step 11: Choose the angle icon in the upper tab. An angle can be created by selecting two lines or three points. Note that the direction in which you choose the objects determines if the acute or the obtuse angle is shown. If your triangle ABC is oriented anticlockwise then choosing B, A, C creates angle at vertex A. Successively selecting C, B, A creates angle at vertex B and finally A, C, B creates angle at vertex C. These are automatically named α, β, γ. The colored circle can be uncolored to remove the angles from being displayed in the graph.

X(1) : Incenter

The incenter is point inside the triangle whose perpendicular distance to each of the sides of the triangle are the same. The incenter is therefore described by the trilinear coordinates 1:1:1, denoting the ratio of the perpendicular distances to BC, CA, and AB respectively. It is the intersection of the angular bisectors of the triangle and forms the center of a circle lying inside the triangle and tangent to all three sides.

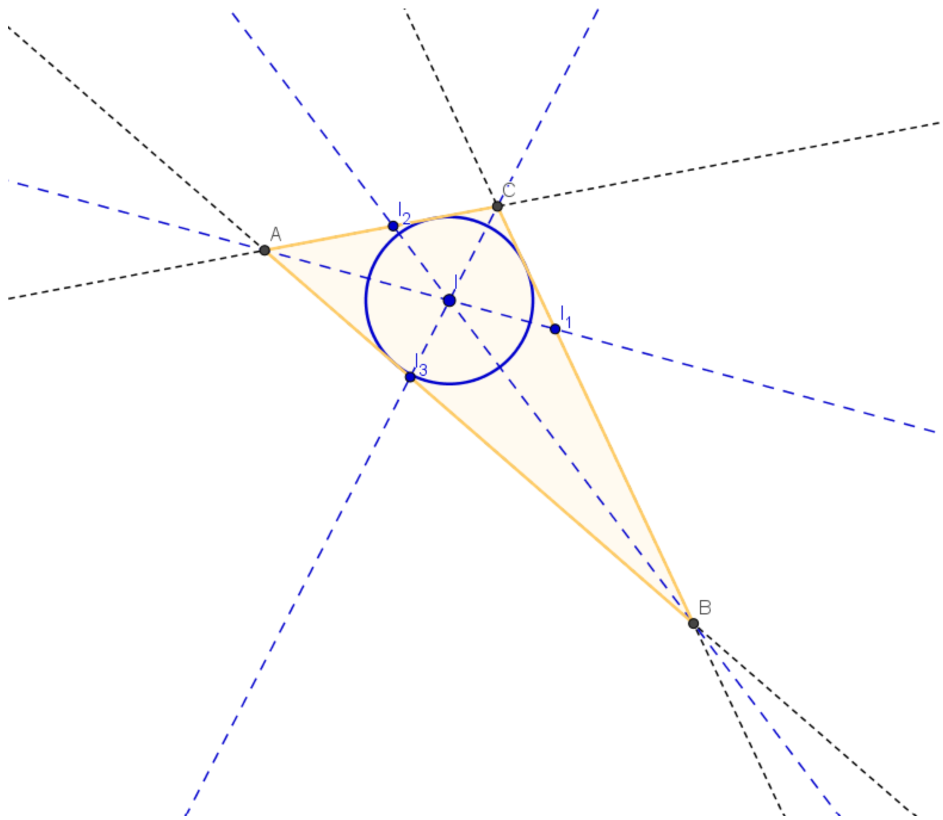

Figure 2: Incenter and incircle of a triangle ABC

Step 1: Choose Angle Bisector. Click the points B, A, C successively to draw the angular bisector through the angle A. Successively selecting C, B, A creates the angular bisector of the angle at vertex B and finally A, C, B creates the angular bisector of the angle at vertex C. These are automatically named f, g and h. The line thickness can be reduced and style changed to dashed if desired.

Step 2: Choose Intersect. Click on the intersection of the three lines. Press Esc to exit from the point creation functionality, then rename the point I.

Step 3: Using Intersect again, find the intersection of f with the line segment BC, g with the line segment CA, and h with the line segment AB. They are named D,E,F by default. Rename them

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I_{1}, I_{2}, I_{3}
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displayed as I_1, I_2, and I_3

Step 4: Deselect f, g, h, I_1, I_2, and I_3 in the algebra view to turn off their visibilty.

Step 5: We now draw the incircle. Select Perpendicular Line. Click any side of the triangle, say AB, then the incenter I.

Step 6: Using the intersect option, find where the perpendicular intersects the side AB. This point is named D by default.

Step 7: Draw the incircle using the Compass option. Choose point D and I. This fixes the radius. Now choose I to fix the center. Rename the object as incircle.

Step 8: Deselect the perpendicular line and the intersection point D in the algebra view.

Step 9: All the objects associated with the incenter are colored uniformly to make it more readable.

Step 10: Choose the Checkbox option. Click in a position on the graph where you would like the checkbox to be displayed. Write an appropriate caption. Click the drop down arrow and choose all objects, one by one, required to be displayed for the incenter and incircle construction: f, g, h, I, I_1, I_2, I_3, and incircle. Click Apply. Press Esc.

X(2) : Circumcenter

The circumcenter is the intersection of the perpendicular bisectors of the sides of the triangle. Its trilinear coordinates is given as 1/a : 1/b : 1/c or bc : ca : ab or csc α: csc β : csc γ, etc.

Step 1: Choose the Perpendicular bisector option. Click the points B, C successively to draw the perpendicular bisector of side BC. Successively selecting C, A creates the perpendicular bisector of side CA and finally A, B creates the perpendicular bisector of side AB. These are automatically named j, k and l. Press Esc to exit the construction mode. The line thickness can be reduced and style changed to dashed if desired.

Step 2: Click on the intersection of the three lines. Press Esc to exit from the point creation functionality, then rename as C'.

Step 3: Find the intersection of the perpendicular bisectors j, k, l with the sides BC, CA, AB of the triangle. Press Esc to exit from the point creation functionality, then rename the points E, F, G as

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C_{1}, C_{2}, C_{3}
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Step 4: Click on the small arrow in the circle button on the upper bar and chose Circle with Center through Point. Click on C' then any of the vertices, say A to create the circumcircle.

Step 5: Note that the circumcircle can also be created by using the Circle through Three Points option and clicking on vertices A, B, C.

Step 6: All the objects associated with the circumcenter are colored uniformly to make it more readable. Here, dark green is chosen.

Step 7: Choose Checkbox option. Click in a position on the graph where you would like the checkbox to be displayed. Write an appropriate caption. Click the drop down arrow and choose all objects required to be displayed for the circumcenter and circumcircle construction: C', C_1, C_2, C_3, j, k, l, circumcircle. Click Apply. Press Esc to exit the Checkbox creation mode.

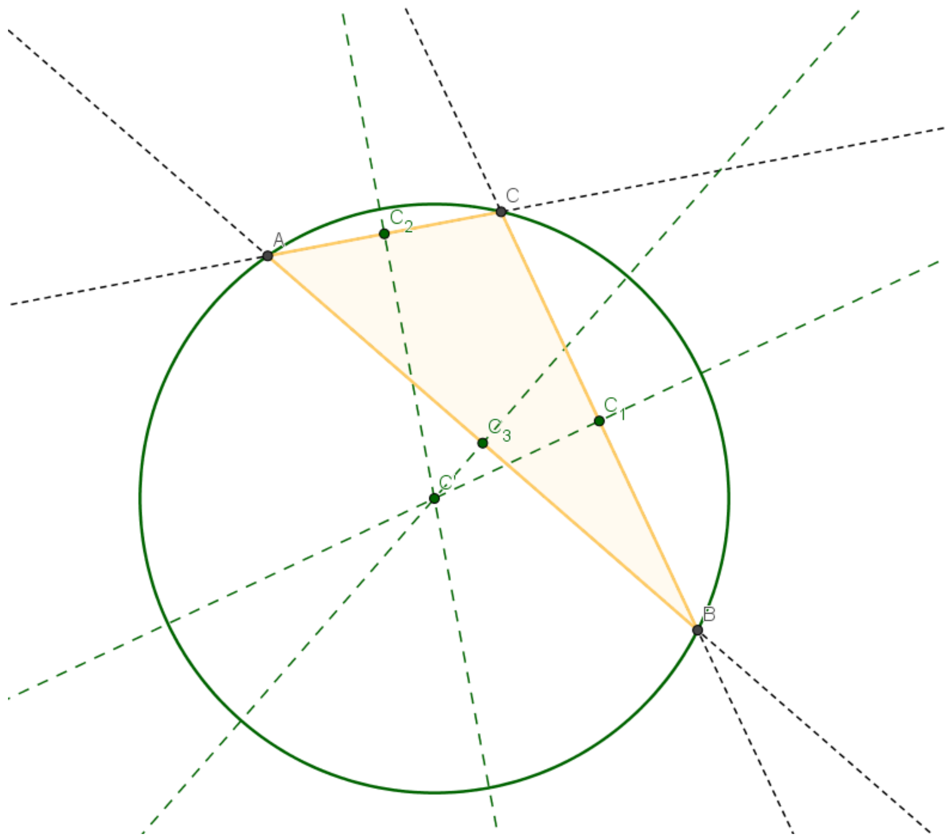


Figure 3: Circumcenter and circumcircle of a triangle ABC

X(3) : Centroid

The centroid is the intersection of the lines through midpoints of the sides of the triangle. Its trilinear coordinates is given as cos A : cos B : cos C or a(b2 + c2 - a2) : b(c2 + a2 - b2) : c(a2 + b2 - c2).

Step 1: Choose the Midpoint option. Click on the endpoints of the segment for which you want to find the midpoint. Click B,C to get midpoint of segment BC. Click C, A to get midpoint of segment CA and click A, B to get midpoint of segment AB. Press Esc then rename the midpoints as M_A, M_B, M_C

Step 2: Choose the line option. Selecting two points will define a line. Choose successively A MA, B M_B, then C M_C. The lines are named n, p, q, by default.

Step 3: Click on the intersection of the three lines. Rename as G.

Step 4: Choose Circle through three points. Click on the points M_A, M_B, M_C. The circle formed is called the Nine Point Circle denoted by X(5) in the Kimberling system.

Step 5: All the objects associated with the centroid are colored uniformly to make it more readable. Here, red is chosen.

Step 6: Choose Checkbox. Click in a position on the graph where you would like the checkbox to be displayed. Write an appropriate caption. Click the drop down arrow and choose all objects required to be displayed for the circumcenter and circumcircle construction: M_A, M_B, M_C, n, p, q, and G. The Nine Point circle may have its own checkbox if desired. Click Apply. Press Esc to exit the Checkbox creation mode.

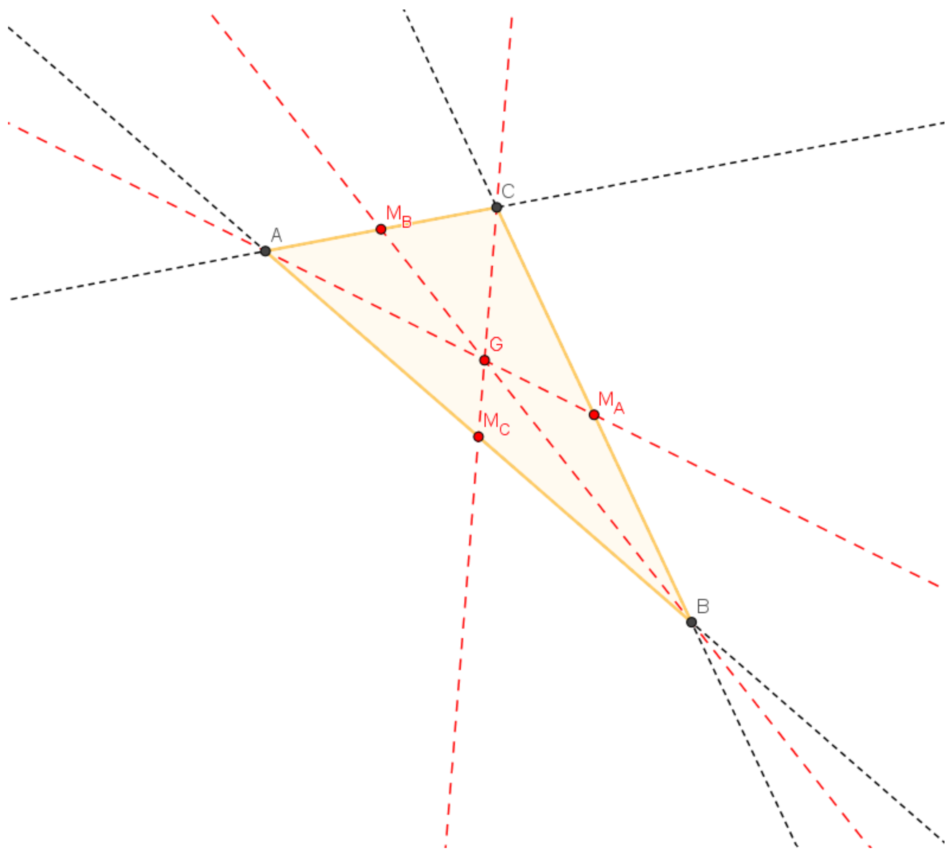


Figure 4: Centroid of a triangle ABC

X(4) : Orthocenter and X(5) : Nine-point Circle

The orthocenter is the intersection of the altitudes of the triangle. Its trilinear coordinates is given as sec A : sec B : sec C, cos A - sin B sin C : cos B - sin C sin A : cos C - sin A sinB, etc.

Step 1: Using Perpendicular Line option, choose the side of the triangle then the opposite vertex. SegBC then A, SegCA then B, and finally SegAB then C. Change line style if desired.

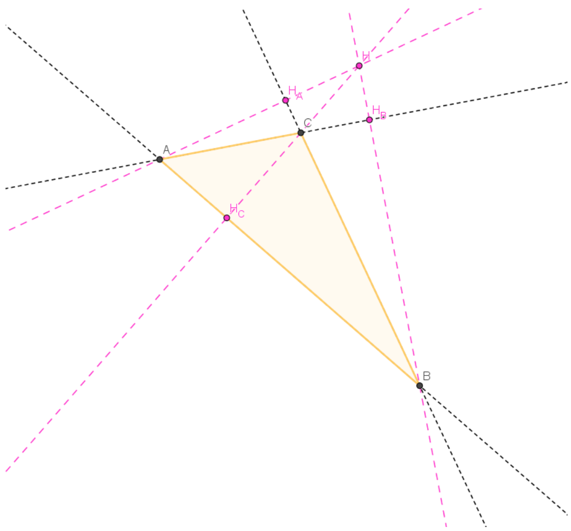
Step 2: Click on the intersection of the three lines. Rename as H.

Step 3: Find the intersections of the altitudes with the lines BC, CA, AB. Rename as H_A, H_B, H_C.

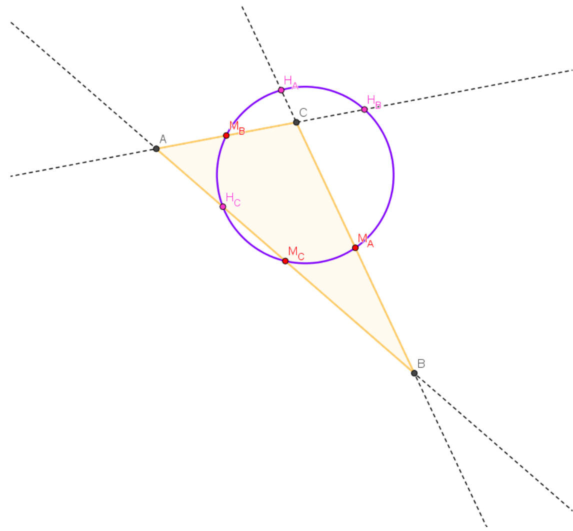
Step 4: Choose Circle through three points. Click on the points M_A, M_B, M_C from the algebra view so that they are shown in the graphing plane. Note that the circle passing through $M_A, M_B, M_C, H_A, H_B, H_C$ is the Nine Point Circle.

Step 5: All the objects associated with the orthocenter are colored uniformly to make it more readable. Here, pink is chosen.

Step 6: Choose Checkbox. Click in a position on the graph where you would like the checkbox to be displayed. Write an appropriate caption. Click the drop down arrow and choose all objects required to be displayed for the orthocenter construction: H_A, H_B, H_C, r, t, f_1, and H. Click Apply. Press Esc to exit the Checkbox creation mode.



(a) Orthocenter and orthocircle of a triangle ABC



(b) Nine-point circle of a triangle ABC

X(6) : Symmedian Point

The symmedian is the intersection of the symmedians, i.e. medians reflected over the corresponding angular bisectors of the triangle. Its trilinear coordinates is given as a : b : c or sin A : sin B : sin C.

Step 1: Choose Checkbox. Click in a position on the graph where you would like the checkbox to be displayed. Write an appropriate caption. Click Apply. Press Esc to exit the Checkbox creation mode.

Step 2: A boolean value is created in the algebra view corresponding to the checkbox created. Rename this variable as z.

Step 3: Rename the angular bisectors drawn while constructing the incenter as m_1, m_2, m_3.

Step 4: Select the angular bisectors by holding down the Ctrl key and clicking the objects in the algebra view. In the object Properties under the Advanced Tab, type "&& z" in addition to the variable already printed. This means that the angular bisectors will be shown for both the incenter and the symmedian point checkboxes.

Step 5: Find the Midpoints of the sides of the triangle by choosing the Midpoint or Center option. Click B,C to get midpoint of segment BC. Click C, A to get midpoint of segment CA and click A, B to get midpoint of segment AB. Press Esc then rename the midpoints as $mm1, mm2, mm3$ Choose the line option. Selecting two points will define a line. Choose successively A-$mm1$, B-$mm2$, then C-$mm3$. Rename the lines as k_1, k_2, k_3.

Step 6: Click on Reflect about Line. Choose the line to be reflected then the line serving as the mirror line. Select k_1 then $mm1$, k_2 then $mm2$, and k_3 then $mm3$. Rename these lines as s1, s2, s3.

Step 7: Plot the point of intersection of the three symmedians s1, s2, s3. Rename as S.

Step 8: All the objects associated with the symmedian point are colored uniformly to make it more readable. Here, pink is chosen.

Step 9: Select the objects $m_1, m_2, m_3, k_1, k_2, k_3$, s1, s2, and s3 in the algebra view. In the object Properties under the Advanced Tab, type "z".

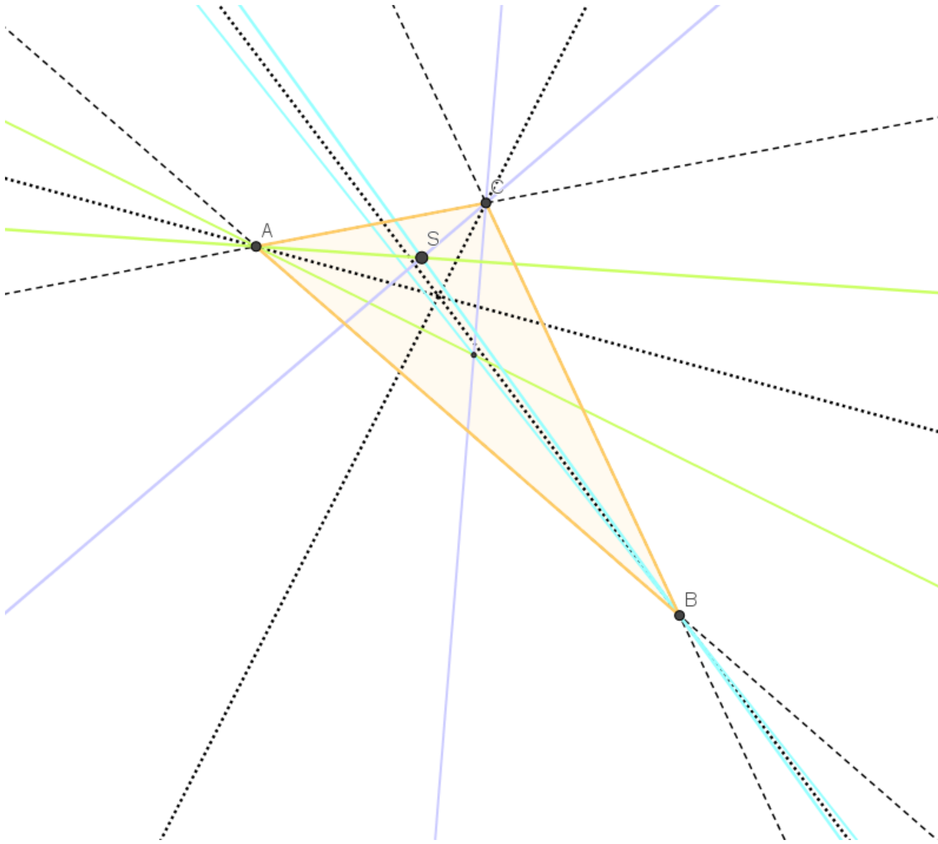


Figure 6: Symmedian Point of a triangle ABC

More X(n) centers

Some Kimberling centers can be derived using other centers. In fact, many centers can be written as a combination of the incenter, circumcenter and centroid with proportional factors including the sidelengths, semiperimeters, radius of the incircle and circumradius.

Step 1: Find the radius of the incircle and circumradius with the following commands:

$$r = \text{Length}(\text{Segment}(I, D))$$
$$R = \text{Length}(\text{Segment}(C', A))$$

Step 2: In the project, the names of the incenter, circumcenter, centroid, and orthocenter are respectively I, C, G and O respectively. Using these position coordinates, which are stored in these variables, derived centers can be plotted by typing the following into the input box.

$X11 = RI - 3rC' + rG$

$X12 = RI + 3rC' - rG$

$X19 = (r + 2R - s)(r + 2R + s)I - 6R(r + 2R)C' - 2(r^2 + 2 * r * R - s^2)G$

$X21 = 3RC' + 2rG$

$X24 = (a^2 - b^2 - c^2)(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)G + 2a^2b^2c^2H$

$X30 = C' - G$

$X31 = (r^2 + s^2)I - 6r * RC' - 4r^2G$

$X32 = -(r^2 + 4r * R - s^2)(r^2 + 2r * R + s^2)I + 6rR(r^2 + 4r * R - s^2)C' + 2r^2(r^2 + 4r * R - 3s^2)G$

$X33 = (r + 2R - s)(r + 2R + s)I - 6r * RC' + 4r * RG$

$X34 = (r + 2R - s)(r + 2R + s)I + 6r * RC' - 4r * RG$

$X37 = (r^2 + 2r * R - s^2)I - 6r * RC' - 2r^2G$

$X38 = (3r^2 + 8rR - s^2)I - 6rRC' - 4r^2G$

$X39 = (r^2 + 4rR - s^2)(r^2 + 2rR + s^2)I - 6rR(r^2 + 4rR - s^2)C' - 2r^2(r^2 + 4rR + s^2)G$

$X40 = I - 2G$

$X44 = (3r^2 + 6rR + s^2)I - 18rRC' - 6r^2G$

$X45 = (3r^2 + 6rR - s^2)I - 18rRC' - 6r^2G$

$X47 = (r^2 - R^2 + s^2)I - 6rRC' - 4r^2G$

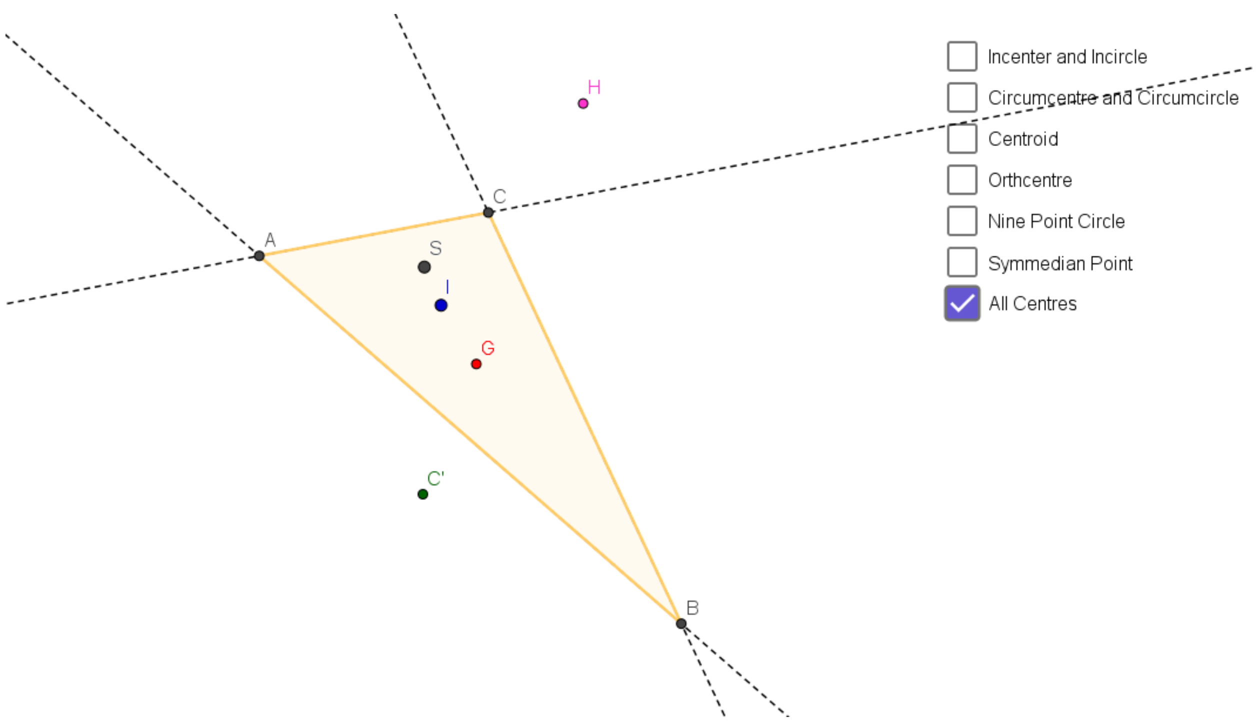


Figure 7: Five centers of a triangle ABC

Experimenting

Try experimenting with the positioning of the centers by:

- Changing positions of vertices A, B and C by dragging the point around the graph.
- Noting differences for obtuse, acute and right triangles.
- Ticking multiple check boxes for displaying multiple centers at once and finding relations
- Finding the radius of incircle (r) and circumcircle (R) and plotting the derived centers mentioned above using the coordinates of the incenter (I), circumcenter (C), centroid (G), and orthocenter (O).

Conclusion

In this project, I have used the mathematical software GeoGebra to give a practical introduction to plotting centers of a triangle. It gives a step by step procedure to create triangles and plot the incenter, circumcenter, centroid, orthocenter, nine-point circle and symmedian point using basic line constructions of angular bisectors, medians, perpendicular bisectors and perpendiculars. More triangle centers can be plotted with formulae involving the centers created in this project and values of radii of incenter and circumcenter. Some of these centers have been mentioned to experiment on if interested. The theory of triangles is ancient but still thriving and GeoGebra is a software that can make its study intuitive and practical.